Can a numerically stable subgrid-scale model for turbulent flow computation be ideally accurate?: A preliminary theoretical study for the Gaussian filtered Navier-Stokes equations

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This paper introduces a candidate for the origin of the numerical instabilities in large eddy simulation repeatedly observed in academic and practical industrial flow computations. Without resorting to any subgrid-scale modeling, but based on a simple assumption regarding the streamwise component of flow velocity, it is shown theoretically that in a channel-flow computation, the application of the Gaussian filtering to the incompressible Navier-Stokes equations yields a numerically unstable term, a cross-derivative term, which is similar to one appearing in the Gaussian filtered Vlasov equation derived by Klimas [J. Comput. Phys. **68**, 202 (1987)] and also to one derived recently by Kobayashi and Shimomura [Phys. Fluids **15**, L29 (2003)] from the tensor-diffusivity subgrid-scale term in a dynamic mixed model. The present result predicts that not only the numerical methods and the subgrid-scale models employed but also only the applied filtering process can be a seed of this numerical instability. An investigation concerning the relationship between the turbulent energy scattering and the unstable term shows that the instability of the term does not necessarily represent the backscatter of kinetic energy which has been considered a possible origin of numerical instabilities in large eddy simulation. The present findings raise the question whether a numerically stable subgrid-scale model can be ideally accurate.

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I. INTRODUCTION

In large eddy simulation (LES) of turbulent flows, where large-scale flow structures are solved directly but small-scale eddies are modeled, numerical instability which leads to nonphysical oscillation or divergence of solution has sometimes been observed. One of the well-known origins of this instability is the dispersive characteristics of the numerical methods used to solve flow equations. In LES, centered finite-difference techniques which involve no artificial viscosity are commonly adopted even in the convection terms in the Navier-Stokes equations. That type of differencing is known to be unstable when the grid resolution is not sufficiently high. In order to overcome this problem, the use of high-order fully conservative schemes [1,2] and the adaptation of high-order filtering (or smoothing) procedures have been attempted [3,4].

Another known cause of this numerical instability is an unstable property of the subgrid-scale (SGS) models adopted. The dynamic Smagorinsky model [5], for instance, can predict both positive and negative eddy viscosities. While the appearance of the negative viscosity allows the description of the backscatter of kinetic energy [the inverse energy cascade in turbulent flows, the occurrence of which has been confirmed [6–11] by *a priori* tests with direct numerical simulation (DNS)], it unfortunately causes instability in the numerical solution achieved using finite-difference methods. This instability problem involved in the dynamic eddy-viscosity-type model has been removed by a smoothing and/or a clipping technique [5,12–14] that mollifies or eliminates the negative values of the eddy-viscosity coefficient. Also, it is known that the tensor-diffusivity model [15–17] behaves unstably in certain situations such as channel flows and turbulent mixing layers where strong shears appear [9,11,17–19]. This instability has been considered to be due to its incorrect near-wall scaling [17] or long-lived negative diffusion taking place near a wall [18]. Recently, in their study regarding a dynamic mixed model [20] Kobayashi and Shimomura [19] have shown theoretically that in the viscous sublayer in a turbulent channel flow, the absolute value of the negative diffusion coefficient derived from the tensor-diffusivity term can be greater than the kinematic viscosity coefficient, resulting in a negative total viscosity and consequently leading to numerical instability.

In the present paper, we perform a theoretical study concerning the Gaussian filtered Navier-Stokes equations and introduce an alternative candidate for the origin of the numerical instability in the LES of near-wall turbulence. The present study is motivated by the filtering problem of the Vlasov-Poisson system, first attempted by Klimas [21,22]. In Ref. [21], to overcome the filamentation problem, he derived a filtered equation for collisionless plasma kinetics by applying the Gaussian filter in the velocity space to the Vlasov-Poisson system. Interestingly, although neither an approximation nor an assumption was made, the resulting formulas have a closed form in terms of the filtered distribution function, that is, the resulting system is solvable without any

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modeling. However, the numerical solution given using this filtered system is, unfortunately, unstable if a numerical method other than a spectral method is employed [23]. This numerical instability is caused by an additional term in the filtered Vlasov-Poisson system, a cross-derivative term yielded by the filtering, the numerical solution of which by, e.g., a finite-difference technique should be unstable and diverge. Because neither modeling nor an approximation is adopted in order to derive the filtered equation, this numerical instability can be considered to arise only from the Gaussian filtering. We show in this paper that a similar cross-derivative term can be found in the filtered Navier-Stokes equations without adapting any SGS model, but with a simple assumption on the streamwise velocity component. The derived cross-derivative term is similar also to that given in the recent study of Kobayashi and Shimomura [19] concerning a mixed SGS model, a study by which we are also motivated. They derived this term from the tensor-diffusivity part of a mixed model, i.e., by adapting a SGS model, while we do not assume a particular SGS model. From this point of view, the present work can be considered a generalization of Kobayashi and Shimomura's work. Furthermore, we point out theoretically that the numerical instability which may be caused by the cross-derivative term does not always correspond to backscatter; namely, the instability we predict is only due to the filtering, but not due to the physical process (backscatter) in turbulent flows.

As we will mention in Sec. IV, the present findings are not simply a numerical problem but raise a question about the feasibility of accurate LES. This numerical problem appears to indicate a performance limitation of current LES.

This paper is organized as follows: In Sec. II, we review and briefly reexamine the work by Klimas. In Sec. III, we perform theoretical investigations regarding the unstable characteristics of the Gaussian filtered Navier-Stokes equations under a wall-bounded flow condition. Section IV provides remarks concerning the feasibility of accurate LES (or a limitation of current LES technique).

II. NUMERICAL INSTABILITY OF THE FILTERED VLASOV-POISSON EQUATION

In 1987, Klimas [21], based on a spectral method, proposed a numerical scheme for the Vlasov-Poisson system,

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}+E(x,t)\frac{\partial f}{\partial v}=0, \tag{1}$$

$$\frac{\partial E}{\partial x}=\int f(x,v,t)dv-1, \tag{2}$$

and

$$\frac{1}{\ell}\int_{x=0}^{x=\ell}\int f(x,v,t)dv\,dx=1,$$

which describes the kinetics of collisionless plasmas in the phase space (x,v). Here, this system is normalized by the plasma frequency, the Debye length, and the thermal velocity. This numerical scheme incorporates a filtering procedure with the Gaussian function in order to mollify the filamentation [21] of the distribution function *f*, which is an infinitely fine structure resulting naturally from the fact that Eqs. (1) and (2) have no diffusion mechanism, and cannot be resolved numerically using finite computer resources. This filtering process, applied only to the velocity space, is carried out by the convolution

$$\overline{F}(x,v,t)=\int_{u=-\infty}^{u=\infty}L(v-u)F(x,u,t)du, \tag{3}$$

where $\overline{(\cdot)}$ denotes the filtered quantity, $L(X)$ is the Gaussian filter function,

$$L(X)=\frac{1}{\Delta\sqrt{2\pi}}\exp\left(-\frac{X^2}{2\Delta^2}\right), \tag{4}$$

which satisfies $\int_{X=-\infty}^{X=\infty}L(X)dX=1$, and Δ is the filter width assumed to be constant throughout this paper. The filtered Vlasov-Poisson system can thus be expressed as

$$\overline{\left(\frac{\partial f}{\partial t}\right)}+\overline{\left(v\frac{\partial f}{\partial x}\right)}+E(x,t)\overline{\left(\frac{\partial f}{\partial v}\right)}=0, \tag{5}$$

$$\frac{\partial E}{\partial x}=\int \overline{f}dv-1. \tag{6}$$

As Klimas showed [21], Eq. (5) can be rewritten into a closed form in terms of $\overline{f}$:

$$\frac{\partial \overline{f}}{\partial t}+v\frac{\partial \overline{f}}{\partial x}+E(x,t)\frac{\partial \overline{f}}{\partial v}=-\Delta^2\frac{\partial^2\overline{f}}{\partial x\partial v}, \tag{7}$$

where the left-hand side of this equation is an advection equation corresponding to Eq. (1) with the replacement $f\rightarrow\overline{f}$, while the term on the right-hand side (RHS) is an additional term yielded through the filtering process. For the convenience of readers, we show concretely the derivation of this additional term. This cross-derivative term is given from the second term of Eq. (5) as follows:

$$\begin{aligned}\overline{\left(v\frac{\partial f}{\partial x}\right)}&=\int L(v-u)u\frac{\partial f}{\partial x}du=-\int (v-u)L(v-u)\frac{\partial f}{\partial x}du\\&\quad+v\int L(v-u)\frac{\partial f}{\partial x}du=-\left[\Delta^2L(v-u)\frac{\partial f}{\partial x}\right]_{-\infty}^{\infty}\\&\quad+\Delta^2\int L(v-u)\frac{\partial^2 f}{\partial x\partial u}du+v\int L(v-u)\frac{\partial f}{\partial x}du\\&=\Delta^2\overline{\left(\frac{\partial^2 f}{\partial x\partial v}\right)}+v\overline{\left(\frac{\partial f}{\partial x}\right)},\end{aligned}$$

and thus,

$$\overline{\left(v\frac{\partial f}{\partial x}\right)}=v\frac{\partial\overline{f}}{\partial x}+\Delta^2\frac{\partial^2\overline{f}}{\partial x\partial v}. \tag{8}$$

Here, we used

$$\overline{\left(\frac{\partial f}{\partial v}\right)}=\frac{\partial \overline{f}}{\partial v}.$$

Unfortunately, the numerical solution of Eq. (7) tends to be unstable when a numerical method, except for the spectral method (such as a finite-difference or a finite-volume technique), is employed [23]. Applying the coordinate transformation of

$$2x=x_1-y_1, \quad 2v=x_1+y_1, \tag{9}$$

the cross-derivative term is rewritten as

$$-\Delta^2\frac{\partial^2\overline{f}}{\partial x\partial v}=\Delta^2\frac{\partial^2\overline{f}}{\partial y_1^2}-\Delta^2\frac{\partial^2\overline{f}}{\partial x_1^2}. \tag{10}$$

The last term of this equation, having a negative coefficient, represents inverse (or retrograde) diffusion, which is known to produce an unstable numerical solution using a finite-difference method or others. This nature of the cross-derivative term leads to numerical instability and a resulting divergence of solution. Note that this instability is not physical but only numerical. The true solution of Eq. (7) should not diverge, because f depending on the pure advection equation (1) is an invariant, and the inequality

$$\min(f)\leqslant\min(\overline{f})<\max(\overline{f})\leqslant\max(f)$$

is always true. (Note that the Gaussian function is positively defined. If a nonpositive defined filter such as the spectral cutoff filter is used, this relation does not always hold.)

We can prove Eq. (10) by a more intuitive manner. First, we express the term in a finite-difference form,

$$\frac{\partial^2\overline{f}(x,v)}{\partial x\partial v}=\lim_{\Delta x\to 0,\Delta v\to 0}\left(\frac{\dfrac{\overline{f}_{1,1}-\overline{f}_{-1,1}}{2\Delta x}-\dfrac{\overline{f}_{1,-1}-\overline{f}_{-1,-1}}{2\Delta x}}{2\Delta v}\right), \tag{11}$$

where

$$\overline{f}_{i,j}\equiv\overline{f}(x+i\Delta x,v+j\Delta v).$$

Consequently, we rewrite this as follows:

$$\frac{\partial^2\overline{f}(x,v)}{\partial x\partial v}=\lim_{\Delta x\to 0,\Delta v\to 0}\left(\frac{\overline{f}_{1,1}-2\overline{f}_{0,0}+\overline{f}_{-1,-1}}{4\Delta x\Delta v}\right)-\lim_{\Delta x\to 0,\Delta v\to 0}\left(\frac{\overline{f}_{-1,1}-2\overline{f}_{0,0}+\overline{f}_{1,-1}}{4\Delta x\Delta v}\right).$$

As a result, we found

$$-\Delta^2\frac{\partial^2\overline{f}}{\partial x\partial v}=-\Delta^2\alpha\frac{\partial^2\overline{f}}{\partial\xi^2}+\Delta^2\alpha\frac{\partial^2\overline{f}}{\partial\varsigma^2}, \tag{12}$$

where ξ and ς are the spatial coordinates parallel to $\overrightarrow{(\Delta x,\Delta v)}$ and $\overrightarrow{(-\Delta x,\Delta v)}$, respectively, and α is a positive constant. This result is equivalent to Eq. (10), and the first term on the RHS of Eq. (12) represents numerically unstable inverse diffusion.

Note that the cross-derivative term was derived without applying any approximation and modeling, meaning that this numerically unstable term is given not by an approximation or modeling, but by the Gaussian filtering only. Meanwhile, the magnitude of this term's coefficient increases with the filter width Δ, revealing that increasing Δ to obtain a smoother profile of f makes the governing equation more unstable.

III. INVESTIGATION OF A NAVIER-STOKES CASE

Filtering approaches have been attempted not only for plasma kinetics analysis but also for turbulent flow computations. In particular, the LES of turbulence has employed various kinds of SGS stress models which are constructed based on a filtering approach and statistical properties of turbulence [11,24]. In this section, we show theoretically that the filtering process may, at least in a certain case, also give rise to numerical instability in flow computations.

The Navier-Stokes equations for incompressible fluid flows are expressed as

$$\frac{\partial u_i}{\partial t}+\frac{\partial u_j u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j} \quad \text{for} \quad i=1,2,3,$$

$$\frac{\partial u_i}{\partial x_i}=0,$$

where Einstein's summation convention is assumed, and $u_i=u_i(x_1,x_2,x_3,t)$ is the velocity component, p is the pressure divided by the constant fluid density, and ν is the kinematic viscosity. Applying a spatial filter to these equations, one obtains

$$\frac{\partial\overline{u_i}}{\partial t}+\frac{\partial\overline{u_j u_i}}{\partial x_j}=-\frac{\partial\overline{p}}{\partial x_i}+\nu\frac{\partial^2\overline{u_i}}{\partial x_j\partial x_j} \tag{13}$$

or

$$\frac{\partial\overline{u_i}}{\partial t}+\frac{\partial\overline{u_j}\,\overline{u_i}}{\partial x_j}=-\frac{\partial\overline{p}}{\partial x_i}+\nu\frac{\partial^2\overline{u_i}}{\partial x_j\partial x_j}-\frac{\partial\tau_{ij}}{\partial x_j}, \tag{14}$$

$$\tau_{ij}\equiv\overline{u_i u_j}-\overline{u_i}\,\overline{u_j}$$

with

$$\frac{\partial\overline{u_i}}{\partial x_i}=0, \tag{15}$$

where τ_{ij} is the so-called SGS stress tensor which generally needs to be modeled.

In the present paper, in order to simplify discussion we assume that the filter is applied only in the x_2 direction. Furthermore, for a while we focus our attention only on one term in Eq. (13),

$$\frac{\partial \overline{u_j u_1}}{\partial x_j} = \frac{\partial \overline{u_1 u_1}}{\partial x_1} + \frac{\partial \overline{u_2 u_1}}{\partial x_2} + \frac{\partial \overline{u_3 u_1}}{\partial x_3} = \overline{\left(u_1 \frac{\partial u_1}{\partial x_1} \right)} + \overline{\left(u_2 \frac{\partial u_1}{\partial x_2} \right)} + \overline{\left(u_3 \frac{\partial u_1}{\partial x_3} \right)}, \tag{16}$$

which is the filtered convection term in the equation for $\overline{u}_1$, and suppose

$$u_1 = \beta x_2 \text{ with } \beta = \beta(x_1, x_3) \tag{17}$$

is true at an instant. The velocity field described by Eq. (17) represents a shear flow such as that forming near a solid wall parallel to the (x_1, x_3) plane and located at $x_2 = 0$. In Ref. [10], which investigates the energy cascade in turbulent wall-bounded flows and analyzes the performance of eddy-viscosity models in the LES, Härtel and Kleiser performed an *a priori* test of near-wall turbulent flows by DNS in order to investigate the energy transfer between the GS and SGS components constructed by filtering DNS data. They have shown that the dissipation of the streamwise component of kinetic energy ($u_1^2/2$ in the present case) provides a dominant contribution to the total dissipation of kinetic energy (see Fig. 6 of Ref. [10]). This is why we pay much attention to term (16). In what follows, we perform explicitly the convolution in Eq. (16) in order to have a closed formula, and then discuss the characteristics of the resulting formula.

Substituting Eq. (17) into Eq. (16) yields

$$\frac{\partial \overline{u_j u_1}}{\partial x_j} = \overline{\left(\beta x_2 \frac{\partial u_1}{\partial x_1} \right)} + \beta \overline{u}_2 + \frac{\partial \beta}{\partial x_3} \overline{(u_3 x_2)}. \tag{18}$$

Recalling Eq. (8), the first term on the RHS is rewritten into a closed form as

$$\overline{\left(\beta x_2 \frac{\partial u_1}{\partial x_1} \right)} = \overline{u}_1 \frac{\partial \overline{u}_1}{\partial x_1} + \Delta^2 \beta \frac{\partial^2 \overline{u}_1}{\partial x_1 \partial x_2},$$

where we use

$$\overline{u}_1 = u_1. \tag{19}$$

Also, the last term of Eq. (18) can be rewritten into a closed form because

$$\overline{(u_3 x_2)} = \Delta^2 \frac{\partial \overline{u}_3}{\partial x_2} + x_2 \overline{u}_3.$$

As a result, we have

$$\frac{\partial \overline{u_j u_1}}{\partial x_j} = \overline{u}_1 \frac{\partial \overline{u}_1}{\partial x_1} + \Delta^2 \beta \frac{\partial^2 \overline{u}_1}{\partial x_1 \partial x_2} + \beta \overline{u}_2 + \frac{\partial \beta}{\partial x_3} \left(\Delta^2 \frac{\partial \overline{u}_3}{\partial x_2} + x_2 \overline{u}_3 \right).$$

Rewriting this as

$$\frac{\partial \overline{u_j u_1}}{\partial x_j} = \overline{u}_1 \frac{\partial \overline{u}_1}{\partial x_1} + \Delta^2 \beta \frac{\partial^2 \overline{u}_1}{\partial x_1 \partial x_2} + \frac{\partial \beta x_2}{\partial x_2} \overline{u}_2 + \Delta^2 \frac{\partial \beta}{\partial x_3} \frac{\partial \overline{u}_3}{\partial x_2} + \frac{\partial \beta x_2}{\partial x_3} \overline{u}_3 = \overline{u}_j \frac{\partial \overline{u}_1}{\partial x_j} + \left(\Delta^2 \beta \frac{\partial^2 \overline{u}_1}{\partial x_1 \partial x_2} + \Delta^2 \frac{\partial \beta}{\partial x_3} \frac{\partial \overline{u}_3}{\partial x_2} \right),$$

the total SGS stress for $\overline{u}_1$ can be represented as

$$\frac{\partial \tau_{j1}}{\partial x_j} = \Delta^2 \beta \frac{\partial^2 \overline{u}_1}{\partial x_1 \partial x_2} + \Delta^2 \frac{\partial \beta}{\partial x_3} \frac{\partial \overline{u}_3}{\partial x_2}. \tag{20}$$

In order to confirm this result, we show here an alternative derivation of Eq. (20). Yeo [25] and others [26] have derived (and extended [27]) a relation equation, which can be applied to any functions *a* and *b*, expressed as

$$\overline{ab} - \overline{a}\overline{b} = \sum_{n=1}^{\infty} \frac{\Delta^{2n}}{n!} \frac{\partial^n \overline{a}}{\partial \chi^n} \frac{\partial^n \overline{b}}{\partial \chi^n}, \tag{21}$$

where $\overline{(\cdot)}$ indicates the Gaussian filtering in the χ direction and χ is an independent variable of *a* and *b*. This equation is very interesting because the RHS terms contain only filtered functions $\overline{a}$ and $\overline{b}$. Using this and assuming $\chi = x_2$, we have

$$\tau_{11} = \Delta^2 \beta \frac{\partial \overline{u}_1}{\partial x_2}, \tag{22}$$

$$\tau_{12} = \Delta^2 \beta \frac{\partial \overline{u}_2}{\partial x_2}, \tag{23}$$

$$\tau_{13} = \Delta^2 \beta \frac{\partial \overline{u}_3}{\partial x_2}. \tag{24}$$

These results combined with Eqs. (15) and (19) recover Eq. (20).

We analyze here characteristics of Eq. (20). Using Eqs. (17) and (19), Eq. (20) can be reconstructed as

$$\frac{\partial \tau_{j1}}{\partial x_j} = \Delta^2 \beta \frac{\partial^2 \overline{u}_1}{\partial x_1 \partial x_2} + U \frac{\partial \overline{u}_1}{\partial x_3}, \tag{25}$$

$$U \equiv \frac{\Delta^2}{x_2} \frac{\partial \overline{u}_3}{\partial x_2}. \tag{26}$$

The last term of Eq. (25) represents the advection of $\overline{u}_1$ in the x_3 direction and hence does not change the amplitude of $\overline{u}_1$. This term may be solved stably using a numerical method for the convection terms with a sufficient grid resolution. The first term on the RHS of the same equation is a cross derivative of $\overline{u_1}$, being quite similar to one appearing in the filtered Vlasov equation (7) and also to one derived by Kobayashi and Shimomura [19] from the tensor-diffusivity term. Namely, this term should be numerically unstable. The coefficient of this term increases as Δ increases, corresponding to the fact that LES tends to be more unstable when a larger grid width is assumed. This coefficient also increases as the absolute value of β increases, implying that a strong shear is more apt to cause numerical instability. [We should note here that the cross-derivative term is unstable in the cases of both $\beta>0$ and $\beta<0$, since the cross-derivative can be decomposed into two diffusion terms having different signs, as shown in Eq. (10).] Actually, numerical instability in LES has frequently been observed in flow computations involving strong shears, as mentioned in Sec. I. The present result is consistent with this fact.

Let us consider the energy dissipation due to the SGS stress (20). The energy transfer for $\overline{u_1}^2/2$ between the GS and the SGS components is described by

$$\varepsilon_1 = \tau_{1j}\frac{\partial \overline{u_1}}{\partial x_j}, \tag{27}$$

whose negative value indicates forward scatter, whereas the positive value indicates backscatter. Substituting Eqs. (22)–(24) into Eq. (27) and rewriting using Eq. (15) yield

$$\varepsilon_1 = \beta\Delta^2\left(\frac{\partial \overline{u_3}}{\partial x_2}\frac{\partial \overline{u_1}}{\partial x_3} - \frac{\partial \overline{u_3}}{\partial x_3}\frac{\partial \overline{u_1}}{\partial x_2}\right). \tag{28}$$

This reveals that ε_1 can be either negative or positive. If, for example, $\partial\overline{u_1}/\partial x_3 = 0$ is true at a certain location, Eq. (28) reduces there to

$$\varepsilon_1 = -(\beta\Delta)^2\frac{\partial \overline{u_3}}{\partial x_3},$$

which is negative when $\partial\overline{u_3}/\partial x_3$ (the spanwise derivative of the spanwise velocity component) is positive, but is positive when $\partial\overline{u_3}/\partial x_3$ is negative.

We provide here a note on assumption (17). Careful readers may think that the velocity profile assumed in Eq. (17) is sufficiently smooth and hence need not be modeled. This natural question can be solved by considering the discussed Gaussian filtering to be the secondary filtering; that is, the velocity u_i is one already smoothed out by a primary filtering. In several SGS models (e.g., the scale similarity [28], the dynamic Smagorinsky [5], and dynamic mixed [29,20,30] models), a twofold filtering is adapted based on the scale invariance of small-scale structures in turbulent flows [11]. In fact, the numerical instabilities in LES have been observed especially when the dynamic Smagorinsky or the tensor-diffusivity models are employed (the latter model does not employ an explicit filter but corresponds to a first-order approximation of the scale-similarity model when the Gaussian filtering is assumed [27]), so that some special treatments for preventing the instability are necessary in these cases [5,9,11–14,17–19].

Finally, we briefly discuss

$$\overline{\left(u_1\frac{\partial u_i}{\partial x_1}\right)} \quad \text{for} \quad i \neq 1,$$

which is a convection term in the filtered Navier-Stokes equations for u_i ($i\neq 1$) in a nonconservative form. Substituting assumption (17) into this, and performing the convolution explicitly or by employing Eq. (21), we obtain

$$\overline{\left(u_1\frac{\partial u_i}{\partial x_1}\right)} = \overline{u}_1\frac{\partial \overline{u_i}}{\partial x_1} + \Delta^2\beta\frac{\partial^2 \overline{u_i}}{\partial x_1 \partial x_2}. \tag{29}$$

The last term of this equation is a cross derivative of the dependent variable and is almost the same as one derived previously. This result shows that the filtered equations for u_i ($i\neq 1$) may also be unstable in the case we have considered.

IV. SUMMARY AND DISCUSSION

The theoretical results given in the preceding section can be summarized as follows: At least in a certain case, the application of the Gaussian filter to the Navier-Stokes equations results in the appearance of a numerically unstable term even though no SGS model is adopted. This unstable term does not always give rise to backscatter of kinetic energy, implying that the numerical instability that the cross-derivative term may cause does not correspond to the physical process, backscatter; this conclusion may explain the numerical results provided by Kobayashi and Shimomura [19] to investigate the signs of the eddy viscosity and of the effective viscosity derived from the tensor-diffusivity term.

The present findings raise the significant question whether a numerically stable SGS model can be an ideal model for describing turbulence, or, in other words, whether a SGS model that can provide ideally accurate results is numerically stable. Our theoretical results indicate that it is in principle possible that a numerically unstable term appears through the filtering process. Namely, if one wants to obtain an accurate LES result under the condition of Eq. (17), he must be faced with the numerically unstable equations (20) and (29), which cannot be solved stably by ready-made finite-difference schemes or others. This difficulty makes a limitation of current LES manifest.

There may be two paths toward overcoming this difficulty: the adaptation of a kind of artificial smoothers, which may violate an accurate solution but has been attempted, and the construction of a stable and accurate solver for the numerically unstable equations. The latter might be extremely difficult to realize, but will be a challenging task.

Furthermore, the present findings may be useful in considering the numerical instabilities of the existing SGS models. For instance, we anticipate that the unstable property of the filtered Navier-Stokes equations may play a role in the LES using the pure scale-similarity model whose unstable property has been considered to result from the model's insufficient diffusivity [9,11,28]. Other instabilities observed in LES might also be interpreted partly based on the present results, though there is no doubt that the instability of the dynamic Smagorinsky model is due to the negative viscosity which accompanies backscatter.

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